1.8 Introduction to Linear Transformation

Def. (Transformation from $\mathbb{R}^n$ to a Subset of $\mathbb{R}^m$)

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \rightarrow T(\vec{x})$$

Rule that assign to each vector $\vec{x} \in \mathbb{R}^n$ another vector $T(\vec{x})$ in $\mathbb{R}^m$.

Definitions

Domain of T : $\mathbb{R}^n$.

Codomain of T : $\mathbb{R}^m$

Range of T : $\{ T(\vec{x}) \text{ in } \mathbb{R}^m : \vec{x} \text{ in } \mathbb{R}^n \}$

Image of $\vec{x}$ under the action of T : $T(\vec{x})$.

Consider

a) Exercise # 33

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow T(\vec{x}) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 \\ 5x_2 \end{bmatrix}$$

b) A modification of # 33.

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow T(\vec{x}) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix}$$

c) Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}$

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow A\vec{x} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Discuss the Shear transformation. (pag 65 4th Edt.)

<u>Definition</u>: LINEAR Transformation.

A transformation

$$T: \text{Domain of } T \longrightarrow \text{Codomain of } T$$

$$\vec{x} \longrightarrow T\vec{x}$$

is linear if

i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, for all $\vec{u}, v$ in Domain of T.

ii) $T(c\vec{u}) = cT(\vec{u})$, for all scalars c and all $\vec{u}$ in the domain of T.

<u>Theorem A</u> If T is a linear transformation, then

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longrightarrow T(\vec{x})$$

i) $T(\vec{0}) = \vec{0}$

ii) $T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

for all vectors $\vec{u}$ and $\vec{v}$ in domain of T and all scalars "c" and "d".

We now show: T in (a) (c) are linear,

but T in (b) is not linear.

In fact, (ask what we want to prove)

(i) For any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$

Want to prove $T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + T\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

<u>Proof.</u>

$$T(\vec{x}+\vec{y}) = \begin{bmatrix} 2(x_1+y_1) - 3(x_2+y_2) \\ x_1+y_1 \\ 5(x_2+y_2) \end{bmatrix} = \begin{bmatrix} 2x_1+2y_1-3x_2-3y_2 \\ x_1+y_1 \\ 5x_2+5y_2 \end{bmatrix}$$

$$\overset{\text{Just reordering}}{=} \begin{bmatrix} (2x_1-3x_2)+(2y_1-3y_2) \\ x_1+y_1 \\ 5x_2+5y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1-3x_2 \\ x_1 \\ 5x_2 \end{bmatrix} + \begin{bmatrix} 2y_1-3y_2 \\ y_1 \\ 5y_2 \end{bmatrix} = T(\vec{x}) + T(\vec{y}) \checkmark$$

Also, we want to prove:

(ii) For any c in $\mathbb{R}$,

$$T(c\vec{x}) = cT(\vec{x})$$

or
$$T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = cT\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Proof:-

$$T(\vec{x}) = T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} = \begin{bmatrix} 2(cx_1) - 3(cx_2) \\ cx_1 \\ 5(cx_2) \end{bmatrix}$$

$$= \begin{bmatrix} c(2x_1 - 3x_2) \\ cx_1 \\ c(5x_2) \end{bmatrix} \longrightarrow = c\begin{bmatrix} 2x_1 - 3x_2 \\ x_1 \\ 5x_2 \end{bmatrix}$$

$$= cT\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = cT(\vec{x}) \checkmark$$

For T defined in (c)

i) $T(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) \overset{\text{properties of matrix multiplication}}{=} A\vec{x} + A\vec{y} = T(\vec{x}) + T(\vec{y}) \checkmark$

$T(c\vec{x}) = A(c\vec{x}) = cA\vec{x} = cT(\vec{x}). \checkmark$

So, any transformation defined from a matrix A as $T(\vec{x}) = A\vec{x}$ is Linear.

They are called "matrix transformation".

For T defined by (b), we can easily prove that is not linear.

For example, the condition (ii) is not satisfied

$$T(c\vec{x}) \neq cT(\vec{x}).$$

In fact,

$$T(c\vec{x}) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1)-3(cx_2) \\ cx_1+4 \\ 5(cx_2) \end{bmatrix}$$

$$= \begin{bmatrix} c(2x_1-3x_2) \\ \boxed{cx_1+4} \\ c(5x_2) \end{bmatrix} \neq \begin{bmatrix} c(2x_1-3x_2) \\ \boxed{c(x_1+4)} \\ c(5x_2) \end{bmatrix}$$

$$= c\begin{bmatrix} x_1-3x_2 \\ x_1+4 \\ 5x_2 \end{bmatrix} = cT(\vec{x})$$

Find the domain and range of T in (a).

Domain $= \mathbb{R}^2$

For the range, we consider

$$T(\vec{x}) = \begin{bmatrix} 2x_1-3x_2 \\ x_1 \\ 5x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ 0 \\ 5x_2 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$$

So range of T = All possible linear combinations of $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$ $= \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}\right\}$

What is the geometrical meaning of

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} \right\} ?$$

Plane in $\mathbb{R}^3$.

Also, Notice that

$$T(\vec{x}) = x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= A\vec{x}$$

where $A = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}$

That means that the transformation in (a) is a "matrix transformation".

It will be proved later that any linear transformation is also a matrix transformation.

Proof: (Thm A) in page ③

$\vec{0}$ in $\mathbb{R}^n$ can be obtained from any $\vec{u} \in \mathbb{R}^n$ multiply by the scalar 0.

$$0\vec{u} = \vec{0}$$

Therefore, $T(0\vec{u}) = T(\vec{0})$

Now, $T(0\vec{u}) = 0T(\vec{u}) \underset{\downarrow}{=} \vec{0}$ in $\mathbb{R}^m$

True because $T(\vec{u}) \in \mathbb{R}^m$ and any vector in $\mathbb{R}^m$ multiply by scalar 0 is $\vec{0}$.

Also,

ii) $T(c\vec{u} + d\vec{v}) \overset{\text{def}}{=} T(c\vec{u}) + T(d\vec{v}) \overset{\text{def}}{=} cT(\vec{u}) + dT(\vec{v})$.

We can easily see that

if $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v}) \Rightarrow T$ is linear

In fact, by choosing $c=1$, and $d=1$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

and by choosing $d=0$ and c arbitrary

$$T(c\vec{u}) = cT(\vec{u}).$$

1.9 The matrix of a Linear Transformation

Reconsider again our transformation in (a).

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 \\ 5x_2 \end{bmatrix} \quad \text{linear}$$

We found

$$T(\vec{x}) = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}.$$

Can any linear transformation be transformed into a matrix transformation?

Yes.

Thm 10

Hip: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation.

Then, there exist a unique matrix A Such that

$$T(\vec{x}) = A\vec{x}, \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

Proof.

If $\vec{x}$ is in $\mathbb{R}^n$, then $\qquad \vec{e}_j = \begin{bmatrix} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

Then, $T(\vec{x}) = T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n)$

$$\overset{\text{linearity (i)}}{=} T(x_1\vec{e}_1) + \cdots + T(x_n\vec{e}_n)$$

$$\overset{\text{linearity (ii)}}{=} x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n)$$

$$= \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}$$

So

$$A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

It's also unique (exercise 33)

Show and discuss Tables 1–4

Def.– a) $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be <u>one-to-one</u> if

for each $\vec{b} \in \mathbb{R}^m$, there exists at most one $\vec{x} \in \mathbb{R}^n$ Such that $T(\vec{x}) = \vec{b}$.

b) T is said to <u>be onto $\mathbb{R}^m$</u> if

for each $\vec{b} \in \mathbb{R}^m$, there exists at least one $\vec{x} \in \mathbb{R}^n$ Such that $T(\vec{x}) = \vec{b}$.

The above definitions are equivalent to the following statements:

a) T is one-to-one if
for each $\vec{b} \in \mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$
has a unique solution or none at all.

b) T is onto $\mathbb{R}^m$ if
for each $\vec{b} \in \mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$
has at least one solution $\vec{x} \in \mathbb{R}^n$